Problem 1.6

A simple capacitor is a device formed by two insulated conductors adjacent to each other. If equal and opposite charges are placed on the conductors, there will be a certain difference of potential between them. The ratio of the magnitude of the charge on one conductor to the magnitude of the potential difference is called the capacitance (in SI units it is measured in farads). Using Gauss's law, calculate the capacitance of

(a) two large, flat, conducting sheets of area A, separated by a small distance d;

(b) two concentric conducting spheres with radii a, b $(b > a)$;

(c) two concentric conducting cylinders of length L, large compared to their radii a, b $(b > a)$.

(d) What is the inner diameter of the outer conductor in an air-filled coaxial cable whose center conductor is a cylindrical wire of diameter 1 mm and whose capacitance is 3×10^{-11} F/m? 3×10^{-12} F/m?

[Replace "concentric" with "coaxial," as these cylinders share a common axis, not a common center.]

Solution

The governing equations of the electric field are Gauss's law and Faraday's law. In the context of electrostatics in vacuum they are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

$$\nabla \times \mathbf{E} = \mathbf{0}.$$

This second equation implies the existence of a potential function $-\Phi$ that satisfies

$$\mathbf{E} = \nabla(-\Phi) = -\nabla\Phi. \tag{2}$$

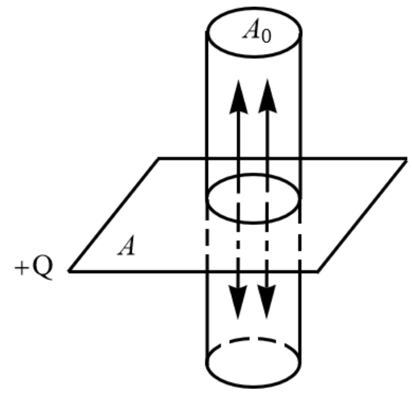
The minus sign is arbitrary mathematically, but physically it indicates that a positive charge in an electric field moves from high-potential regions to low-potential regions (and vice-versa for a negative charge). Assuming that empty space separates the conductors, the capacitance is

$$C = \frac{Q}{\Phi}.$$

Note that Φ is interpreted as the work it takes to move a positive unit charge from the low-potential conductor to the high-potential conductor.

Part (a)

Consider the flat conducting sheet with charge $+Q$ and integrate both sides of equation (1) over the cylindrical volume V_0 shown below. The black arrows are the electric field vectors, and they point vertically because otherwise charges would accelerate along the sheet.



Equation (1) becomes

$$\iiint_{V_0} \nabla \cdot \mathbf{E}\, dV = \iiint_{V_0} \frac{\rho}{\epsilon_0}\, dV.$$

Using the divergence theorem on the left side results in the integral form of Gauss's law.

$$\oiint_{S_0} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \iiint_{V_0} \rho\, dV$$

The volume integral on the right is the total charge enclosed; since the charge is evenly distributed on the sheet, the total enclosed charge is σA_0, where $\sigma = Q/A$.

$$\oiint_{S_0} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \left(\frac{Q}{A} A_0 \right)$$

The closed surface integral on the left consists of three open surfaces: the top circle S_1, the bottom circle S_2, and the lateral cylindrical surface S_3.

$$\iint_{S_1} \mathbf{E} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{E} \cdot d\mathbf{S} + \iint_{S_3} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \left(\frac{Q}{A} A_0 \right)$$

The electric field vectors are completely vertical, so $\mathbf{E} \cdot d\mathbf{S} = EA_0$ on S_1 and $\mathbf{E} \cdot d\mathbf{S} = EA_0$ on S_2 and $\mathbf{E} \cdot d\mathbf{S} = 0$ on S_3.

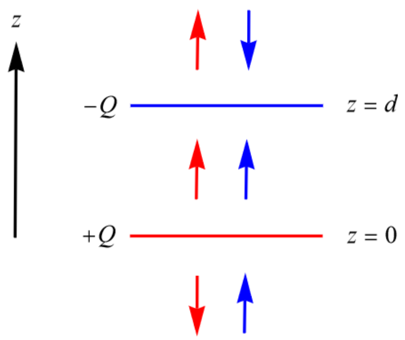
$$EA_0 + EA_0 + 0 = \frac{1}{\epsilon_0} \left(\frac{Q}{A} A_0 \right)$$

Solve for E, the electric field magnitude.

$$2EA_0 = \frac{QA_0}{\epsilon_0 A}$$

$$E = \frac{Q}{2\epsilon_0 A}$$

To find the electric field when there are two flat sheets with equal and opposite charges, use the principle of superposition.



Add the electric fields from each sheet vectorially to get the total field.

$$\mathbf{E} = \begin{cases} \left(\dfrac{Q}{2\epsilon_0 A} + \dfrac{-Q}{2\epsilon_0 A}\right)\hat{\mathbf{z}} & \text{if } z > d \\ \left(\dfrac{Q}{2\epsilon_0 A} + \dfrac{Q}{2\epsilon_0 A}\right)\hat{\mathbf{z}} & \text{if } 0 < z < d \\ \left(\dfrac{-Q}{2\epsilon_0 A} + \dfrac{Q}{2\epsilon_0 A}\right)\hat{\mathbf{z}} & \text{if } z < 0 \end{cases} = \begin{cases} \mathbf{0} & \text{if } z > d \\ \dfrac{Q}{\epsilon_0 A}\hat{\mathbf{z}} & \text{if } 0 < z < d \\ \mathbf{0} & \text{if } z < 0 \end{cases}$$

Since what we need is the potential difference between the sheets, we consider the z-component of equation (2) for $0 < z < d$.

$$\mathbf{E} = -\nabla\Phi \quad \rightarrow \quad E_z = -\frac{d\Phi}{dz} \quad \rightarrow \quad \Phi(z) = -\int E_z \, dz$$

Φ is the work required to move a positive unit charge from the (low-potential) $-Q$ sheet to the (high-potential) $+Q$ sheet, so the limits of integration go from d to 0.

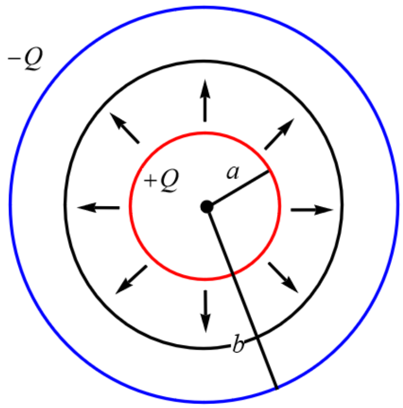
$$\Phi = -\int_d^0 \frac{Q}{\epsilon_0 A}\, dz = \frac{Q}{\epsilon_0 A}\int_0^d dz = \frac{Q}{\epsilon_0 A}(d)$$

Therefore, the capacitance of two large, flat, conducting sheets of area A, separated by a small distance d is

$$C = \frac{Q}{\Phi} = \frac{Q}{\frac{Q}{\epsilon_0 A}(d)} = \frac{\epsilon_0 A}{d}.$$

Part (b)

In order to find the electric field between two concentric conducting spheres with radii a and b and charges $+Q$ and $-Q$, respectively,



integrate both sides of equation (1) over the volume of a concentric sphere with radius r, where $a < r < b$. The black arrows indicate the electric field vectors.

$$\iiint\limits_{x^2+y^2+z^2\leq r^2} \nabla \cdot \mathbf{E}\, dV = \iiint\limits_{x^2+y^2+z^2\leq r^2} \frac{\rho}{\epsilon_0}\, dV$$

Using the divergence theorem on the left side results in the integral form of Gauss's law.

$$\oiint\limits_{x^2+y^2+z^2=r^2} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \iiint\limits_{x^2+y^2+z^2\leq r^2} \rho\, dV$$

The volume integral on the right is the total charge enclosed by the Gaussian surface.

$$\oiint\limits_{x^2+y^2+z^2=r^2} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0}(Q)$$

Because of the symmetry with respect to the origin, the electric field is entirely radial: $\mathbf{E} = E_r \hat{\mathbf{r}}$.

$$\oiint\limits_{x^2+y^2+z^2=r^2} E_r \hat{\mathbf{r}} \cdot (\hat{\mathbf{r}}\, dS) = \frac{1}{\epsilon_0}(Q)$$

E_r is constant on the Gaussian surface $x^2 + y^2 + z^2 = r^2$, so it can be pulled in front of the integral.

$$E_r \oiint\limits_{x^2+y^2+z^2=r^2} dS = \frac{1}{\epsilon_0}(Q)$$

Solve for E_r, the electric field magnitude.

$$E_r(4\pi r^2) = \frac{Q}{\epsilon_0}$$

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2}$$

Since what we need is the potential difference between the spheres, we consider the r-component of equation (2) for $a < r < b$.

$$\mathbf{E} = -\nabla\Phi \quad \rightarrow \quad E_r = -\frac{d\Phi}{dr} \quad \rightarrow \quad \Phi(r) = -\int E_r\,dr$$

Φ is the work required to move a positive unit charge from the (low-potential) $-Q$ sphere to the (high-potential) $+Q$ sphere, so the limits of integration go from b to a.

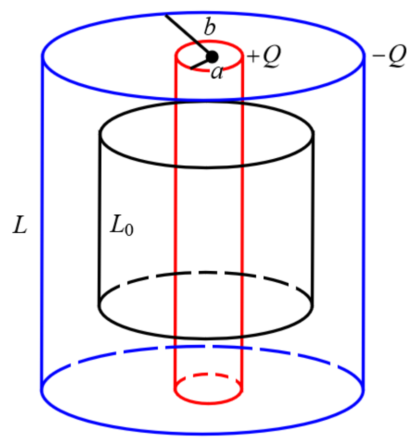
$$\begin{aligned}\Phi = -\int_b^a \frac{Q}{4\pi\epsilon_0 r^2}\,dr = \frac{Q}{4\pi\epsilon_0}\int_b^a \left(-\frac{1}{r^2}\right) dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r}\right)\bigg|_b^a &= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) \\ &= \frac{Q}{4\pi\epsilon_0}\left(\frac{b-a}{ab}\right)\end{aligned}$$

Therefore, the capacitance of two concentric conducting spheres with radii a, b $(b > a)$ is

$$C = \frac{Q}{\Phi} = \frac{Q}{\frac{Q}{4\pi\epsilon_0}\left(\frac{b-a}{ab}\right)} = \frac{4\pi\epsilon_0 ab}{b-a}.$$

Part (c)

In order to find the electric field between two coaxial conducting cylinders with radii a and b and charges $+Q$ and $-Q$, respectively,



integrate both sides of equation (1) over the volume V_0 of a coaxial cylinder with radius r, where $a < r < b$, and length L_0. The electric field vectors point radially outward because otherwise charges would accelerate in the cylinders.

$$\iiint_{V_0} \nabla\cdot\mathbf{E}\,dV = \iiint_{V_0} \frac{\rho}{\epsilon_0}\,dV$$

Using the divergence theorem on the left side results in the integral form of Gauss's law.

$$\oiint_{S_0} \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\epsilon_0}\iiint_{V_0} \rho\,dV$$

The volume integral on the right is the total charge enclosed; since the charge is evenly distributed on the cylinder, the total enclosed charge is $\sigma(2\pi a L_0)$, where $\sigma = Q/(2\pi a L)$.

$$\oiint_{S_0} \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\epsilon_0}\left(\frac{Q}{L}L_0\right)$$

Because of the symmetry with respect to the cylinders' axis, the electric field is entirely radial: $\mathbf{E} = E_r\hat{\mathbf{r}}$.

$$\oiint_{S_0} E_r\hat{\mathbf{r}}\cdot(\hat{\mathbf{r}}\,dS) = \frac{1}{\epsilon_0}\left(\frac{Q}{L}L_0\right)$$

E_r is constant on the Gaussian surface S_0, so it can be pulled in front of the integral.

$$E_r\oiint_{S_0} dS = \frac{1}{\epsilon_0}\left(\frac{Q}{L}L_0\right)$$

Solve for E_r, the electric field magnitude.

$$E_r(2\pi r L_0) = \frac{QL_0}{\epsilon_0 L}$$

$$E_r = \frac{Q}{2\pi\epsilon_0 rL}$$

Since what we need is the potential difference between the cylinders, we consider the r-component of equation (2) for $a < r < b$.

$$\mathbf{E} = -\nabla\Phi \quad\to\quad E_r = -\frac{d\Phi}{dr} \quad\to\quad \Phi(r) = -\int E_r\,dr$$

Φ is the work required to move a positive unit charge from the (low-potential) $-Q$ cylinder to the (high-potential) $+Q$ cylinder, so the limits of integration go from b to a.

$$\begin{aligned}\Phi = -\int_b^a \frac{Q}{2\pi\epsilon_0 rL}\,dr = \frac{Q}{2\pi\epsilon_0 L}\int_a^b\frac{dr}{r} = \frac{Q}{2\pi\epsilon_0 L}(\ln r)\bigg|_a^b &= \frac{Q}{2\pi\epsilon_0 L}(\ln b - \ln a)\\ &= \frac{Q}{2\pi\epsilon_0 L}\ln\frac{b}{a}\end{aligned}$$

Therefore, the capacitance of two coaxial conducting cylinders of length L, large compared to their radii a, b $(b > a)$ is

$$C = \frac{Q}{\Phi} = \frac{Q}{\frac{Q}{2\pi\epsilon_0 L}\ln\frac{b}{a}} = \frac{2\pi\epsilon_0 L}{\ln\frac{b}{a}}.$$

Part (d)

Since the coaxial cable is filled with air, the potential difference between the cylinders is less than it would be if there were empty space (vacuum) by a factor of κ. κ is called the dielectric constant, and the capacitance calculated in part (c) changes to

$$C = \frac{Q}{\frac{\Phi}{\kappa}} = \kappa\left(\frac{Q}{\Phi}\right) = \frac{2\pi\epsilon_0\kappa L}{\ln\frac{b}{a}}.$$

The problem is asking for an inner diameter, so change the radii to diameters.

$$C = \frac{2\pi\epsilon_0\kappa L}{\ln\frac{2b}{2a}} = \frac{2\pi\epsilon_0\kappa L}{\ln\frac{D_b}{D_a}}$$

Solve for D_b, the inner diameter of the outer conductor.

$$\ln\frac{D_b}{D_a} = \frac{2\pi\epsilon_0\kappa L}{C}$$

$$\frac{D_b}{D_a} = \exp\left(\frac{2\pi\epsilon_0\kappa L}{C}\right)$$

$$D_b = D_a\exp\left(\frac{2\pi\epsilon_0\kappa L}{C}\right)$$

Assuming a reasonable temperature and pressure of 20°C and 1 atm, respectively, the dielectric constant is $\kappa \approx 1.00059$. If the capacitance per unit length is $C/L = 3\times10^{-11}$ F/m, then

$$D_b = D_a\exp\left(\frac{2\pi\epsilon_0\kappa L}{C}\right) \approx (1\text{ mm})\exp\left[\frac{2\pi(8.854\times10^{-12})(1.00059)}{3\times10^{-11}}\right] \approx 6\text{ mm}.$$

If the capacitance per unit length is $C/L = 3\times10^{-12}$ F/m instead, then

$$D_b = D_a\exp\left(\frac{2\pi\epsilon_0\kappa L}{C}\right) \approx (1\text{ mm})\exp\left[\frac{2\pi(8.854\times10^{-12})(1.00059)}{3\times10^{-12}}\right] \approx 1\times10^{8}\text{ mm}.$$